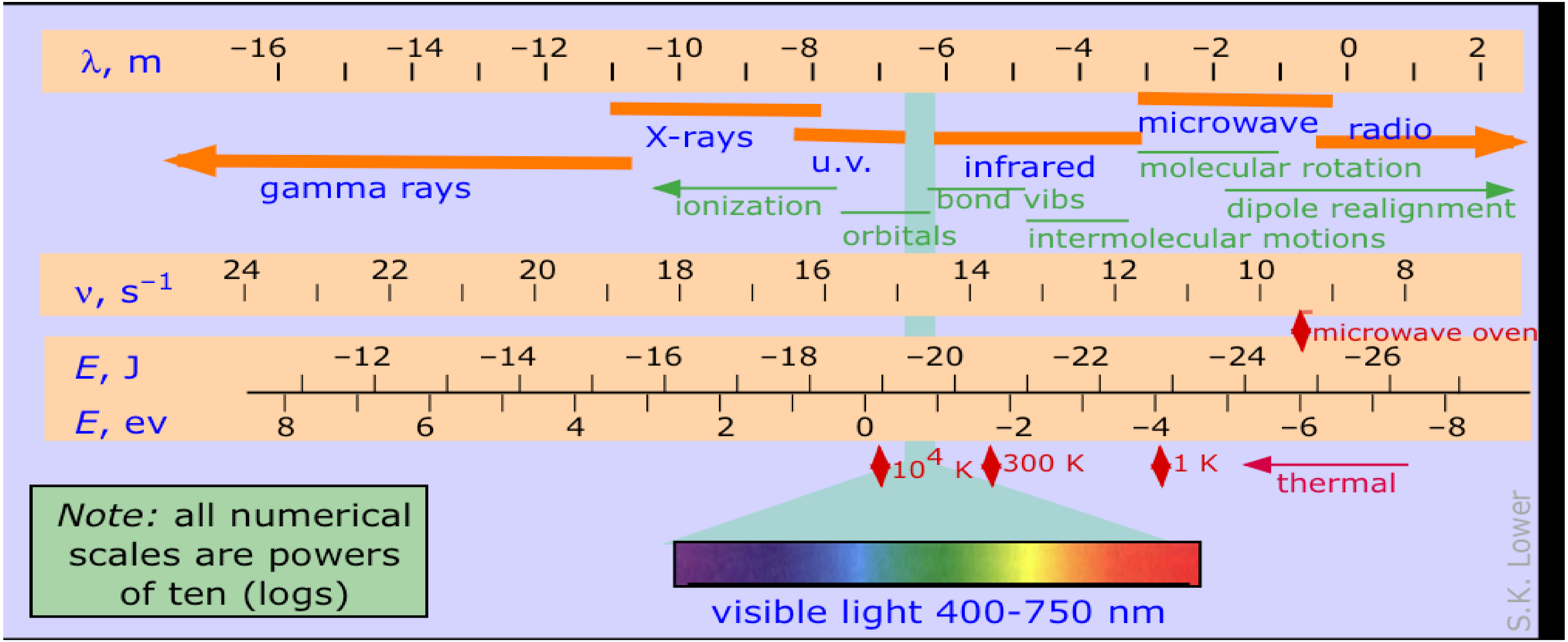
Everything You Always Wanted to Know about Radiative transfer

But Were Afraid to Ask

The spectrum corresponds to waves of various wavelengths and frequencies.

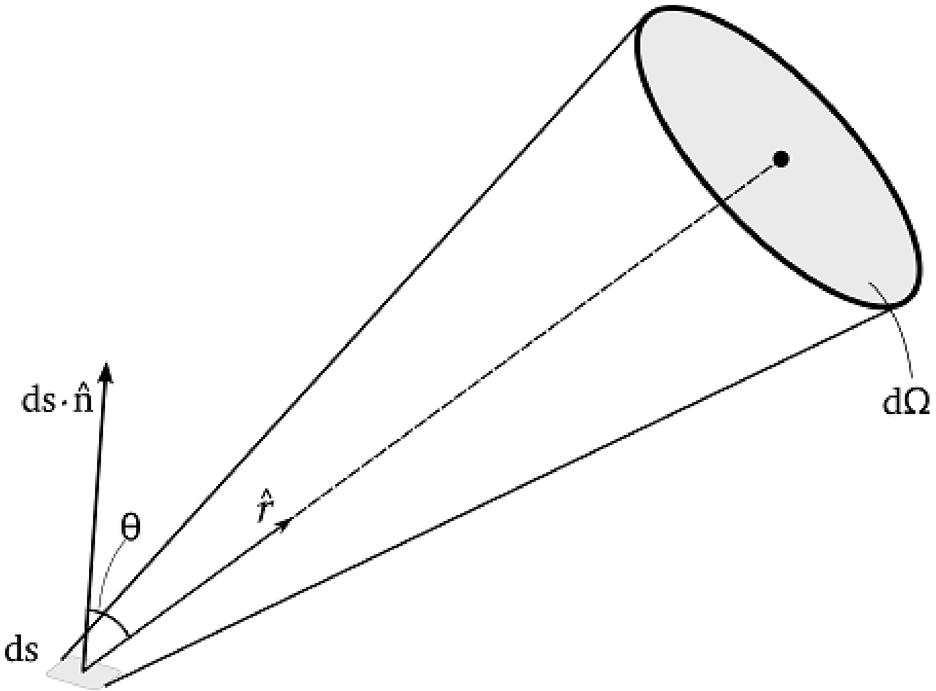
We can divide the spectrum up into various regions, as is done in the next Slide



Energy is expressed in terms of **Frequency** and **Absolute Temperature**

E=kT, where k=1.38.10^{-16} erg/K (*Boltzmann's constant*)

and **E=hν**, where h=6.425.10^{-17} erg/K (*Planks Constant*)



Specific Intensity I_ν (or **I_λ**): It is the **same as flux ($dE_\nu/dt/ds$)** <u>except</u> limited to those photons headed in a particular direction (i.e. **confined within a certain solid angle dΩ**):

$$dE_\nu/dt = I_\nu \, ds \cos\theta \, d\Omega \, d\nu$$

Note: The **Specific Intensity, $I_\nu(\mathbf{x}, t; \mathbf{r})$**, is a scalar function of four variables (ν or λ plus position **x**, time, t, and **r** which is a unit vector with the direction and sense the line of propagation from the observer to the source (see Figure $ds \cdot \mathbf{n} \cdot \mathbf{r} = ds \cdot \cos\theta$, since **n** and **r** are both **unit vectors**).

UNITS

Specific Intensity **I_ν :** $ergs.s^{-1}\ cm^{-2}\ ster^{-1}\ Hz^{-1}$

Specific Intensity **I_λ :** $ergs.s^{-1}\ cm^{-2}\ ster^{-1}\ cm^{-1}$

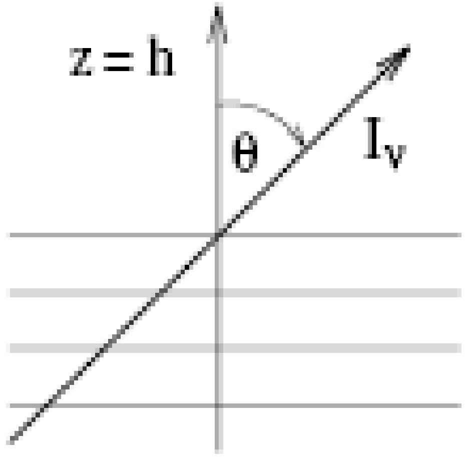
Differential Flux

$F_\nu = I_\nu \cos\theta \, d\Omega$: $ergs.s^{-1}\ cm^{-2}\ Hz^{-1}$

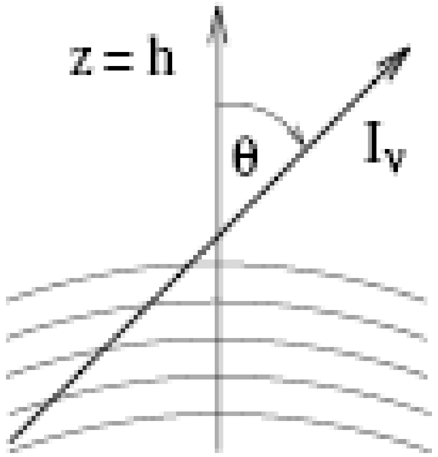
The **plane-parallel** medium approximation

From this point onwards it is convenient to write: <u>cosθ=μ</u>.

The **Specific Intensity , $\mathbf{I_\nu}$(x**, *t* ; **r)**, is a scalar function of four variables. When the radiation field is constant or slowly varying (*we may drop t from the parenthesis*), and if we have axial symmetry with the z-axis along the axis of symmetry the **Specific Intensity**, **$\mathbf{I_\nu}$(x**, *t* ; **r)**, simplifies to: **$\mathbf{I_\nu}$(z**; θ**)**, where θ is the angle between **z** and **r** (direction of the line of propagation from the observer to the source).



This is the case of vertical stratification or "plane parallel layers"; they often represent a local approximation to the curved shells of spherical objects such as the Sun. At the center of the Solar Disk θ=0, at the limb 90^0.



More About Specific Intensity ($\mathbf{I_\nu}$) (a)

$\mathbf{I_\nu}$ is the monochromatic intensity; the total intensity is:

$$I=\int_{\infty}^{0} I_\nu d\nu = \int_{0}^{\infty} I_\lambda d\lambda$$

$$I_\lambda = -I_\nu \frac{d\nu}{d\lambda} = I_\nu \frac{c}{\lambda^2} = I_\nu \frac{\nu^2}{c}$$

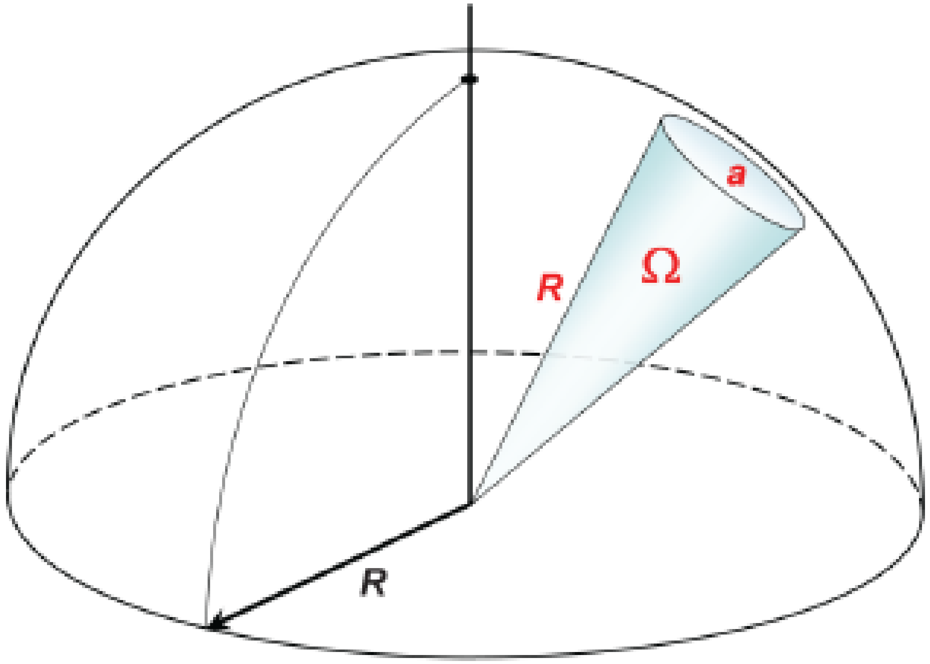
***Exercise**: Verify the unit conversion from I_ν to I_λ.*

The Second equation permits us to switch from frequency to wavelength if necessary.

Since photons are the basic carrier of electro-magnetic interactions, intensity is the basic macroscopic quantity to use in formulating radiative transfer. In particular, the definition per steradian ensures that the intensity along a ray in vacuum does not diminish with travel distance

Solid Angle steradian

- Solid Angle Ω: 2D angle in 3D space
- Measures how large the object appears to an observer
- Solid angle is expressed in a dimensionless unit called a steradian (sr)
- Full sphere is 4π sr



Remember

$\Omega = a/R^2$

and

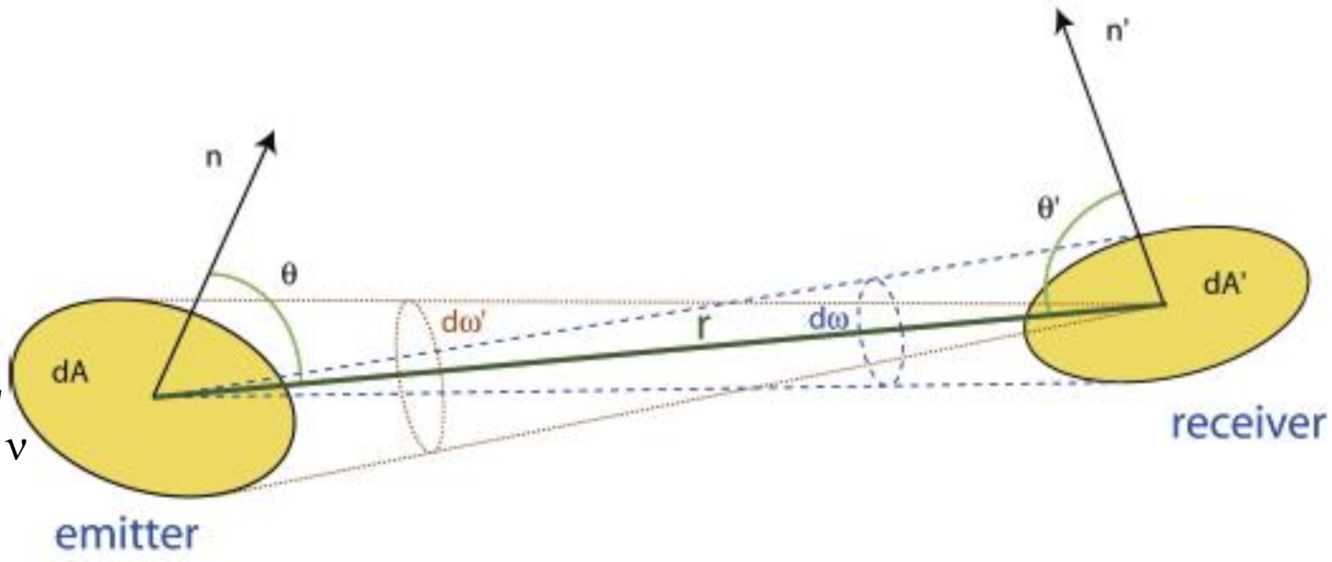
$d\Omega = da/R^2$

More About Specific Intensity (I_ν)

Invariance of the specific intensity

The area element **dA** emits radiation towards **dA'**. In the absence of any matter between emitter and receiver (no absorption and emission on the light paths between the surface elements) the amount of energy emitted and received through each surface elements is:

$$\left.\begin{aligned}
dE_\nu &= I_\nu dA\cos\theta d\omega d\nu dt \\
dE'_\nu &= I'_\nu dA'\cos\theta' d\omega' d\nu' dt \\
dE_\nu &= dE'_\nu \\
d\omega &= dA'\cos(\theta') / r^2 \\
d\omega' &= dA\cos(\theta) / r^2
\end{aligned}\right\} \Rightarrow I_\nu = I'_\nu$$

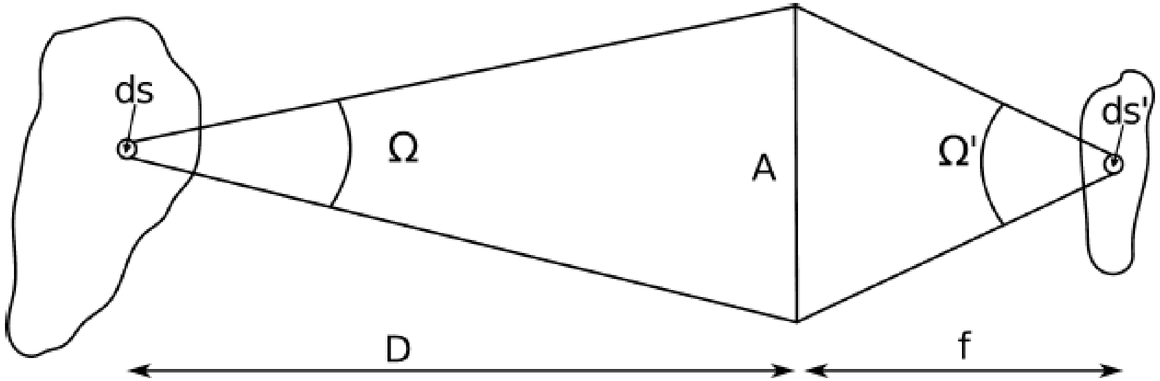


More About Specific Intensity (I_ν)

Example: Invariance of the specific intensity

Consider an extended source (angular width is greater than the angular resolution of our telescope) of surface **s** where a part **ds** is observed by a telescope of aperture **A**. Let the image on the focal plane (at f) be **ds'**. The energy received on **A** will be equal to the energy received on **ds'** for a lossless instrument (see figure). We have, setting cosθ=1 for simplicity:

$$\left.\begin{aligned} dE_\nu &= I_\nu ds \Omega d\nu dt \\ dE_\nu &= I'_\nu ds' \Omega' d\nu' dt \\ I_\nu ds \Omega &= I'_\nu ds' \Omega' \\ \Omega &= A / D^2 \\ \Omega' &= A / f^2 \\ ds / ds' &= D^2 / f^2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} & I_\nu = I'_\nu \text{ and} \\ & \frac{dE_\nu}{ds d\nu dt} = I_\nu \Omega' = I_\nu \frac{A}{f^2} \end{aligned}\right\}$$



From Specific Intensity ($\mathbf{I_\nu}$) we may define the following:

Net Flux ($\mathbf{F_\nu}$ in erg $s^{-1}cm^{-2}Hz^{-1}$) and mean intensity averaged over all directions($\mathbf{J_\nu}$ in erg $cm^{-2}s^{-1}Hz^{-1}ster^{-1}$, just as for $\mathbf{I_\nu}$). Remember: $\cos\theta=\mu$.

In axial symmetry with the z-axis ($\theta = 0$) along the axis of symmetry we have: $\mathbf{d\Omega=2\pi.\sin\theta d\theta}$.

For isotropic Radiation Field we have $\mathbf{J_\nu} = \mathbf{I_\nu}$ and $\mathbf{F_\nu} = 0$.

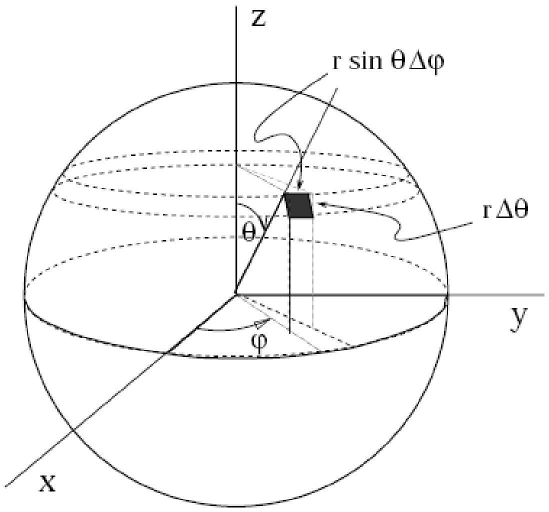
$$J_\nu = \int_{4\pi} I_\nu d\Omega =$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\nu(z,\theta)\ \sin\theta d\theta d\varphi =$$

$$= -\frac{1}{2}\int_0^{\pi} I_\nu(z,\theta)\ d\cos\theta = \frac{1}{2}\int_{-1}^{1} I_\nu(z,\theta)\ d\mu$$

$$F_\nu = \int_{4\pi} I_\nu \cos\theta\ d\Omega$$

$$F_\nu = \int_{-1}^{1} I_\nu(z,\theta)\cdot\mu\cdot d\mu$$

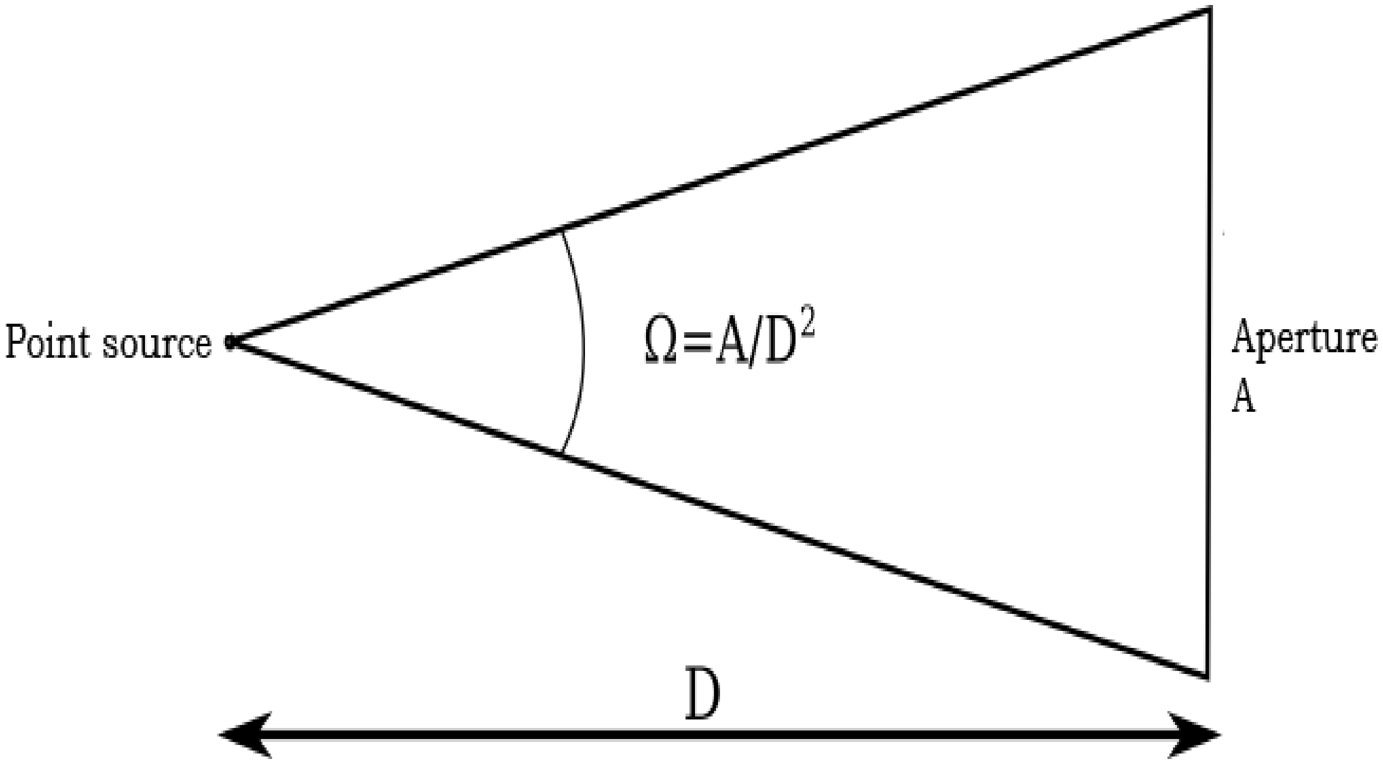


More About Net Flux (F_ν)

Example: Net Flux of Point source

Absolute Luminocity: $L_\nu = 4\pi R^2 F_\nu$

Energy Received by Aperture: $E_\nu = L_\nu \frac{\Omega}{4\pi} = \frac{L_\nu}{4\pi} \frac{A}{D^2} = A \frac{R^2}{D^2} F_\nu$



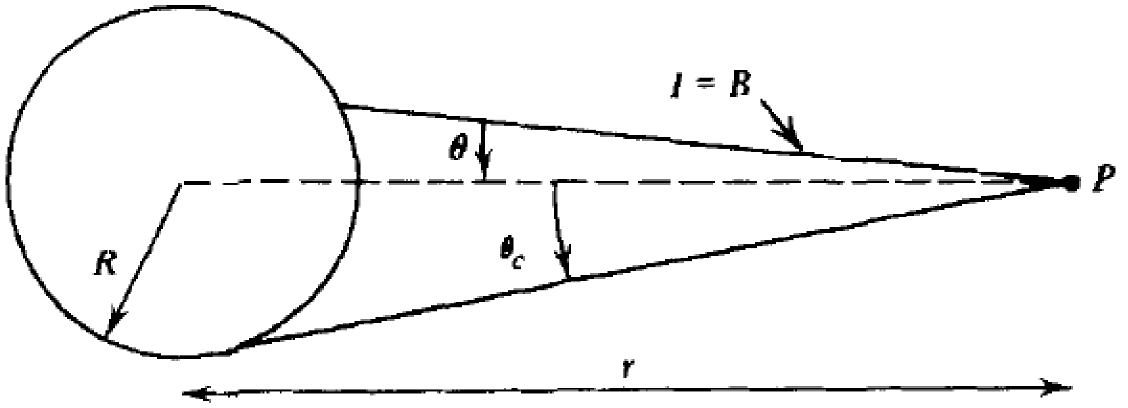
L_ν is the total amount of energy radiated by a star of radius **R** per unit of time within a solid angle of 4π sterad.
A telescope of aperture **A**, at distance **D** from the star, will receive the fraction of this energy within a solid angle **$\Omega = A/D^2$**.

More About Net Flux (F_ν)

Example: Net Flux from a Uniformly bright sphere

Remember: In axial symmetry: $\mathbf{d\Omega=2\pi.\sin\theta d\theta}$.

Let us calculate the flux at an arbitrary distance from a sphere of uniform brightness (specific intensity) $\mathbf{B=I_\nu}$; the sphere is an isotropic source. At P, the specific intensity is $\mathbf{I_\nu}$ (or $\mathbf{B}$) and we have:



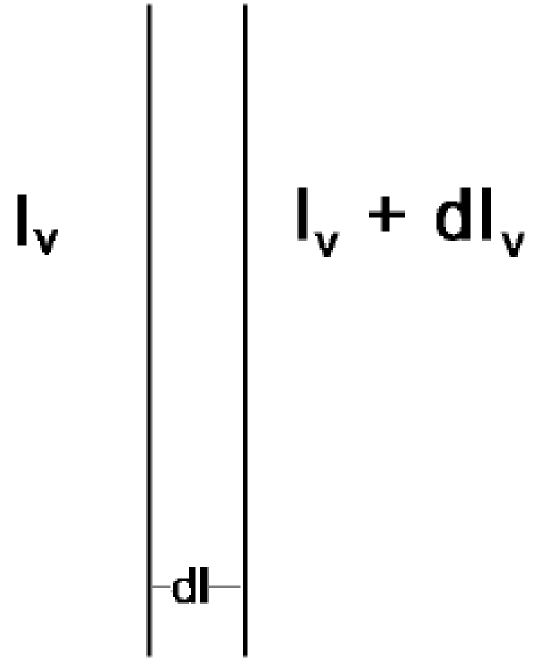
$$\frac{E_\nu}{ds d\nu dt} = \int I_\nu \cos\theta d\Omega = I_\nu \int_0^{2\pi} d\varphi \int_0^{\theta_c} d\theta \sin\theta\cos\theta =$$

$$= 2\pi I_\nu \int_{\cos\theta_c}^{1} \mu d\mu = \pi I_\nu \left(\frac{R}{r}\right)^2$$

Radiative Transfer Equation (a)

If a ray passes through matter, energy may be added or subtracted from it by ***emission*** or ***absorption***. The specific intensity will not in general remain constant. The emission coefficient, $\mathbf{j_\nu}$,(measured in ergs.s^{-1}·ster^{-1}· Hz^{-1}g^{-1}) represents the energy locally added to the radiation $\mathbf{dI_\nu = j_\nu \rho \cdot dl}$, where **l** is the geometrical path length along the beam in cm**.**

Note: Sometimes you may see $\mathbf{dI_\nu = j_\nu \cdot dl}$, $\mathbf{j_\nu}$ units: ergs.s^{-1}cm^{-3}ster$^{(-1)}$Hz^{-1} (ρ is included into $\mathbf{j_\nu}$ and the emission coefficient will be the product $\mathbf{j_\nu \cdot \rho}$)



Radiative Transfer Equation (b)

If a ray passes through matter, energy may be added or subtracted from it by ***emission*** or ***absorption***. The specific intensity will not in general remain constant. "***Scattering***" of photons into and out of the beam can also affect the intensity. We define the absorption coefficient, $\mathbf{k_\nu}$ ($cm^2.g^{-1}$), by the following equation : $\mathbf{dI_\nu = -k_\nu \cdot \rho \cdot I_\nu \cdot dl}$, representing the loss of intensity in a beam as it travels a distance **dl**.

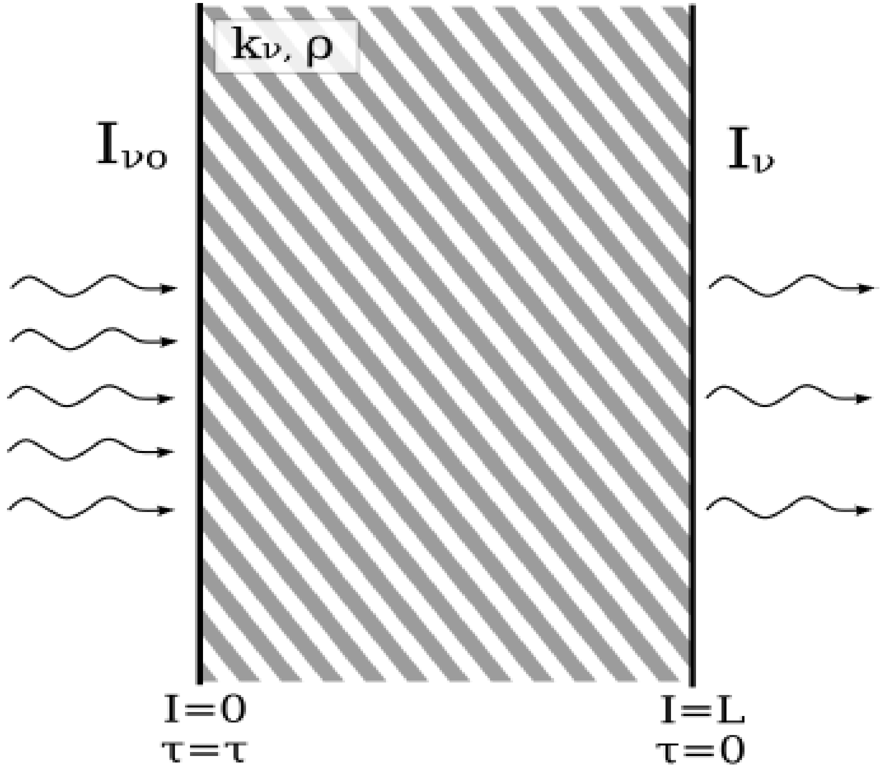
Note that: $\mathbf{k_\nu = \kappa_\nu + \sigma_\nu}$

$\boldsymbol{\kappa_\nu}$: is extinction & $\boldsymbol{\sigma_\nu}$: is Scattering

Some times we write: $\mathbf{\alpha_\nu n = k_\nu \rho}$

(cross section times number density)

You may see $\mathbf{dI_\nu = -k_\nu \cdot I_\nu \cdot dl}$, with $\mathbf{k_\nu}$ units: cm^{-1} (ρ is included into $\mathbf{k_\nu}$; the absorption coefficient will be the product $\mathbf{k_\nu \cdot \rho}$)



Radiative Transfer Equation (c)

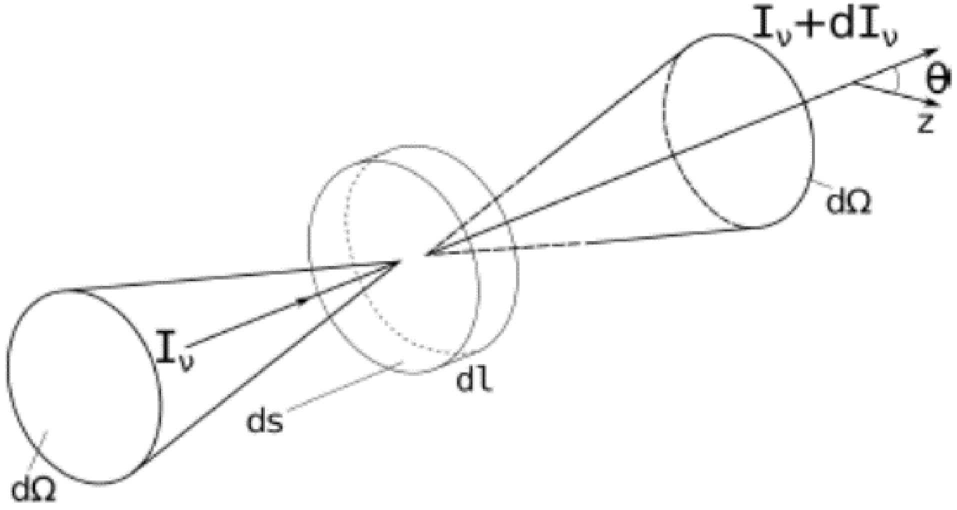
Putting all (mostly the previous two slides) together we have:

$$dI_\nu ds d\Omega d\nu dt = j_\nu \rho dl ds d\Omega d\nu dt - k_\nu \rho I_\nu dl ds d\Omega d\nu dt \Leftrightarrow \frac{dI_\nu}{dl} = j_\nu \rho - k_\nu \rho I_\nu$$

This equation demonstrates that the energy received by the observer within the solid angle dΩ will change due to emission and absorption from the material within the cylinder (See Fig). For θ being the angle between the vertical **z** and the line of propagation from the observer to the source we have **dl·cosθ= dl·μ= dz**, and we have:

$$\left(\frac{\mu}{\rho}\right)\frac{dI_\nu}{dz} = j_\nu - k_\nu I_\nu \text{ setting } d\tau_\nu = -k_\nu \rho dz \Rightarrow$$

$$\Rightarrow \mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \frac{j_\nu}{k_\nu} = I_\nu - S_\nu$$



Radiative Transfer Equation (d)

We have seen in the previous slide the Radiative Transfer Equation in terms of specific intensity, optical depth and the source function $\mathbf{S_\nu}$. This basic equation expresses that photons do not decay spontaneously so that the intensity along a ray does not change unless photons are added to the beam or taken from it; without such processes, intensity is invariant along rays.

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

$$\left(\frac{\mu}{\rho}\right)\frac{dI_\nu}{dz} = j_\nu - k_\nu I_\nu$$

The optical depth $\mathbf{d\tau_\nu = -k_\nu \rho dz}$ (also monochromatic optical path length) is a dimensionless quantity that is measured along the beam across a layer of geometrical thickness **dz**. The RTE takes a particularly simple form if we replace **dz** with optical depth, $\mathbf{d\tau_\nu}$.

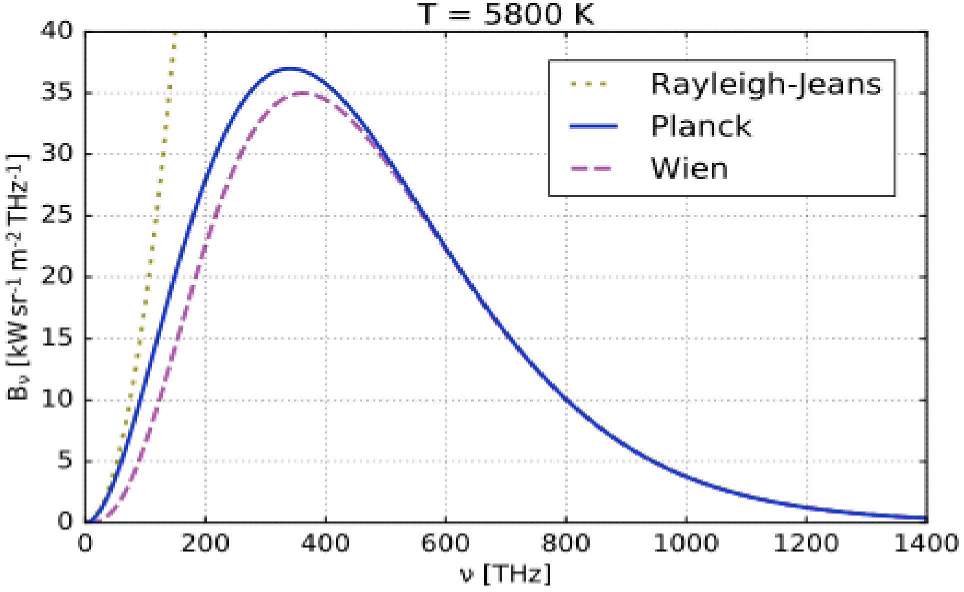
The source function $\mathbf{S_\nu}$ is equal to the Plank Function under Local Thermodynamic Equilibrium conditions.

$$B_\nu(T) = \left(\frac{2h\nu^3}{c^2}\right)\frac{1}{\exp\left(\frac{h\nu}{kT}\right)-1} \quad \text{(Plank Function)}$$

Radiative Transfer Equation (e): About the Source Function

We saw that the source function **S_ν** is equal to the Plank Function under Local Thermodynamic Equilibrium conditions. It has the same units as the Specific Intensity **I_ν :** ergs.s^{-1} cm^{-2} ster^{-1} Hz^{-1}

$$S_\nu = B_\nu(T) = \left(\frac{2h\nu^3}{c^2}\right)\frac{1}{\exp\left(\frac{h\nu}{kT}\right)-1} \quad \text{(Plank Function)}$$



A simplified source function **S_ν** may be derived in the limiting case $h\nu \ll 1$.

$$S_\nu = B_\nu = \frac{2kT}{\lambda^2}$$

This simplified source function is the **Rayleigh–Jeans** approximation to **S_ν** and applies to wavelengths in the range of radio waves

Radiative Transfer Equation (e'):

Rayleigh–Jeans approximation and the brightness temperature.

Brightness temperature: A measure of radiation in terms of the **temperature** of a hypothetical blackbody emitting an identical amount of radiation at the same wavelength. The **brightness temperature** is obtained by applying the inverse of the Planck function to the measured radiation intensity $\mathbf{I_\nu}$; in other words by solving $\mathbf{I_\nu} = \mathbf{B_\nu(T_b)}$ for $\mathbf{T_b}$.

The Rayleigh–Jeans approximation, $\mathbf{I_\nu = 2kT_b/\lambda^2}$,provides an easier inversion.

***Exercise**: Derive the Rayleigh-Jeans approximation from the Plank Function*

Radiative Transfer Equation (f):
Formal Solution for semi-infinite atmosphere

The equation of transfer is a linear differential equation, which implies that a formal solution exists for the radiation field in terms of the source function. In this case we will integrate the equation from the position of the observer at $\boldsymbol{\tau_\nu}$=0, to infinite "depth" (*semi-infinite atmosphere*).

$$\left.\begin{matrix} \mu \dfrac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \Leftrightarrow \dfrac{dI_\nu}{d\tau_\nu} - I_\nu \dfrac{1}{\mu} = -S_\nu \dfrac{1}{\mu} \\ \text{Integrating Factor: } \exp\left(-\dfrac{1}{\mu}\tau_\nu\right) \end{matrix}\right\} \Rightarrow I_\nu\left(\tau_\nu = 0, \mu\right) = \int_0^\infty S_\nu \exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu}$$

***Exercise**: Provide the intermediate steps. What is the formal solution if the atmosphere is not semi-infinite but extends from 0 to optical depth* $\boldsymbol{\tau_\nu}$?

The solution shows the result of contribution to the intensity observed by emission along the line of sight ∞ to 0, decreased at each point by the exponential factor $\mathbf{e^{-t/\mu}}$.

Radiative Transfer Equation (f'):
Formal Solution for semi-infinite atmosphere

Using the Rayleigh–Jeans approximation in the position of the source function we may write the equation of transfer (*semi-infinite atmosphere*) in terms of the **brightness temperature** (**T** is a function of $\boldsymbol{\tau_\nu}$):

$$\left.\begin{aligned} I_\nu(\tau_\nu = 0, \mu) &= \int_0^\infty S_\nu \exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu} \\ I_\nu(\tau_\nu = 0, \mu) &= \frac{2kT_b}{\lambda^2} \\ S_\nu &= \frac{2kT}{\lambda^2} \end{aligned}\right\} \Rightarrow \begin{cases} \dfrac{2kT_b}{\lambda^2} = \displaystyle\int_0^\infty \frac{2kT}{\lambda^2}\exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu} \\ T_b = \displaystyle\int_0^\infty T \cdot \exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu} \end{cases}$$

Radiative Transfer Equation (g):
The Eddington-Barbier approximation

The source function $\mathbf{S_\nu(t,\mu)}$, where **t** stands for optical depth may be Taylor-expanded about $\mathbf{t=\mu}$: $\mathbf{S_\nu(t,\mu)=S_\nu(t=\mu,\mu)+a(t-\mu)}$**+higher order terms** (which are dropped). This result is substituted in the solution of the radiation transfer equation:

$$I_\nu\left(\tau_\nu=0,\mu\right)=\int_0^\infty S_\nu \exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu}\simeq S_{\nu,t=\mu}\int_0^\infty \exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu}+a\int_0^\infty t\cdot\exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu}-a\mu\int_0^\infty \exp\left(-\frac{t}{\mu}\right)\frac{dt}{\mu}\Rightarrow$$

$$\Rightarrow I_\nu\left(\tau_\nu=0,\mu\right)\simeq S_{\nu,t=\mu}$$

The Eddington-Barbier approximation implies that the brightness (specific intensity) equals the Source function at optical depth $\boldsymbol{\mu}$. Therefore at the center of the solar disk observe optical depth $\boldsymbol{\mu=1}$ while at the limb the optical depth decreases, $\boldsymbol{\mu\to 0}$ hence we observe upper layers of the solar atmosphere.

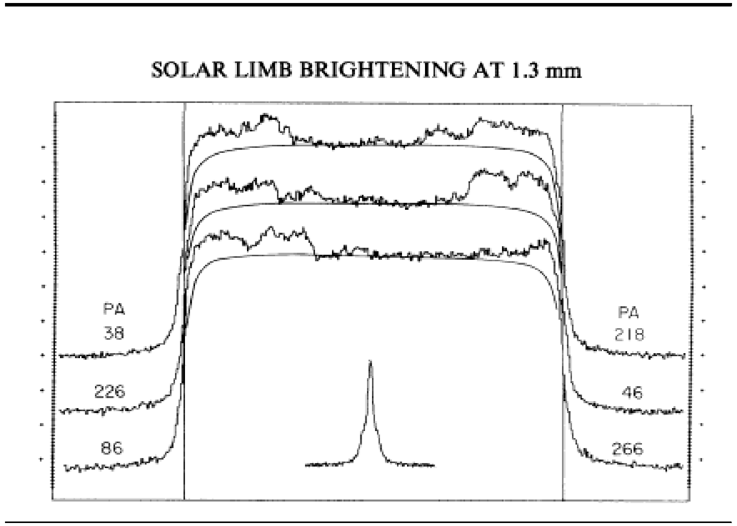
Radiative Transfer Equation (g'):

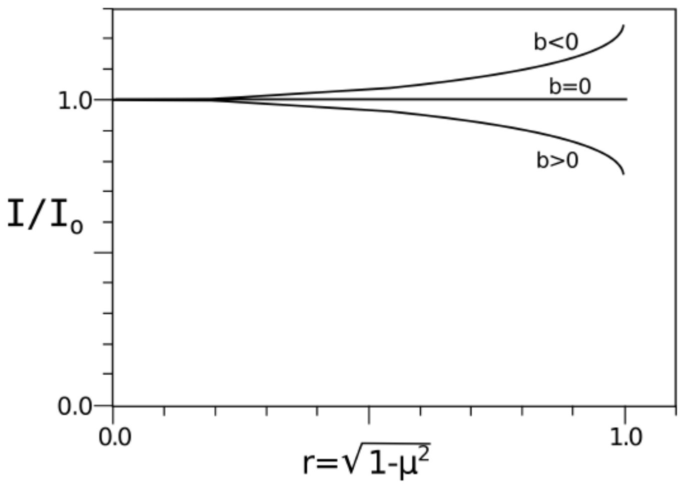
The Eddington-Barbier approximation: Limb Darkening & Brightening on the Solar Disk

From the Taylor expansion, in the previous slide, $\mathbf{S_\nu(t,\mu) \approx S_\nu(t=\mu,\mu)+a(t-\mu)}$, we may write the source function: $\mathbf{S_\nu(t) \approx A+Bt}$. Substituting in the solution of the radiation transfer equation we have: $\mathbf{I_\nu(0,\mu)=A+B\mu}$
From this we have $\mathbf{I_\nu(0,1)=A+B}$, disk center and:

***Exercise**: Provide the intermediate steps.*

$$\frac{I_\nu(0,\mu)}{I_\nu(0,1)} = \frac{A+B\mu}{A+B}$$





Radiative Transfer Equation (h):

Constant Source Function S_ν.

We start from the formal Solution to the Radiative Transfer Equation for an atmosphere of finite depth $\boldsymbol{\tau_\nu}$ when the Source Function $\mathbf{S_\nu}$ is constant:

$$\left.\begin{array}{l} \mu \dfrac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \\[2ex] \text{Int. Factor: } \exp\left(-\dfrac{1}{\mu}\tau_\nu\right) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} I_\nu(\tau_\nu=0,\mu) = I_\nu(\tau_\nu,\mu)e^{-\frac{\tau_\nu}{\mu}} + \displaystyle\int_0^{\tau_\nu} S_\nu e^{-\frac{t}{\mu}} \frac{dt}{\mu} \\[2ex] I_\nu(\tau_\nu=0,\mu) = I_\nu(\tau_\nu,\mu)e^{-\frac{\tau_\nu}{\mu}} + S_\nu\left(1-e^{-\tau_\nu}\right) \end{array}\right.$$

Exercise: *Verify the Solution of the Transfer Equation for finite depth*

Radiative Transfer Equation (h'):

Constant Source Function S_ν (limiting cases τ_ν<<1 and τ_ν>>1)

- In the equations from the previous Slide we drop the μ for simplicity.
- The first term of the Solution represents an external source of radiation under the layer; this affects the result in the case of τ_ν->0 which corresponds to optically thin (transparent) medium.
- The case τ_ν->∞ represents the optically thick (opaque) medium approximation.

$$I_{\nu,0}=I_{\nu=\tau_\nu}e^{-\tau_\nu}+S_\nu\left(1-e^{-\tau_\nu}\right)$$

$$I_{\nu,0}=S_\nu\left(1-e^{-\tau_\nu}\right)\approx\tau_\nu S_\nu,\ \tau_\nu\ll 1\ \text{(transparent)}$$

$$I_{\nu,0}=S_\nu\left(1-e^{-\tau_\nu}\right)\approx S_\nu,\ \tau_\nu\ll 1\ \text{(opaque)}$$

Exercise: *Provide the intermediate steps*

Radiative Transfer Equation (h”):

Constant Source Function $\mathbf{S_\nu}$ (limiting cases τ_ν<<1 and τ_ν>>1)

- In an optically thick layer (τ_ν>>1) of constant **T** the Specific Intensity $\mathbf{I_\nu}$ equals the Plank Function. Further more the brightness temperature $\mathbf{T_b}$ equals **T**. From brightness temperature measurements in the metric radio emission we may verify that the coronal temperature is ~10^6K. On the other hand, transient radio emission (bursts) is characterized by a brightness temperature ~10^8K, which implies emission mechanisms of non thermal origin.
- In an optically thin layer (τ_ν<<0) of constant **T** the brightness temperature $\mathbf{T_b}$ equals τ_ν **T**.