Introduction to plasma

Part 03: Magneto-hydrodynamics, MHD Waves

MHD Waves (I)

When studying <u>small amplitude</u> waves we have to <u>linearize</u> the MHD equations around an equilibrium state (where all variables are constant, and they are denoted by a subscript o). Next we consider all MHD variables as sums of the equilibrium value plus small perturbations (denoted by a subscript 1). In the process we retain only first order terms. We start from a set of MHD equations (we have eliminated **j** from the Amber Law):

$$\left.\begin{aligned}
&\text{Mass Conservation: } \frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \mathbf{v}) = 0\\
&\text{Momentum: } \rho_m \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \frac{\nabla\times\mathbf{B}}{\mu_o}\times\mathbf{B} - \nabla p\\
&\text{Induction Equation } R_L \gg 1\text{: } \frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})\\
&\text{State: } \frac{d}{dt}\left(\frac{p}{\rho_m^{\gamma}}\right) = 0,\ \text{we set: } \rho_m = \rho_{mo} + \rho_{m1},\\
&\mathbf{v} = \mathbf{v_1}, \mathbf{B} = \mathbf{B_o} + \mathbf{B_1}, p = p_o + p_1
\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}
&\frac{\partial \rho_{m1}}{\partial t} + \rho_{mo}\nabla\mathbf{v}_1 = 0\\
&\rho_{mo}\frac{\partial \mathbf{v}_1}{\partial t} = \frac{\nabla\times\mathbf{B_1}}{\mu_o}\times\mathbf{B_o} - \nabla p_1\\
&\frac{\partial \mathbf{B_1}}{\partial t} = \nabla\times(\mathbf{v}_1\times\mathbf{B_o})\\
&\frac{dp_1}{dt} - c_s^2\frac{d\rho_{m1}}{dt} = 0\\
&\text{where: } c_s^2 = \frac{\gamma p_o}{\rho_{mo}}
\end{aligned}\right.$$

MHD Waves (II)

In the linearized MHD equations we seek the solutions in the harmonic form for all perturbations ~exp(i(**k**.**r**-ωt)). We have:

$$\left.\begin{aligned} &\frac{\partial\rho_{m1}}{\partial t}+\rho_{mo}\nabla\mathbf{v}_1=0 \\ &\rho_{mo}\frac{\partial\mathbf{v}_1}{\partial t}=\frac{\nabla\times\mathbf{B_1}}{\mu_o}\times\mathbf{B_o}-\nabla p_1 \\ &\frac{\partial\mathbf{B_1}}{\partial t}=\nabla\times\left(\mathbf{v}_1\times\mathbf{B_o}\right) \\ &\frac{dp_1}{dt}\text{-}c_s^2\frac{d\rho_{m1}}{dt}=0,\ \text{where: } c_s^2=\frac{\gamma p_o}{\rho_{mo}} \end{aligned}\right\}\Rightarrow\left\{\begin{aligned} &\text{-}\omega\rho_{m1}+\rho_{mo}\mathbf{k}\cdot\mathbf{v}_1=0 \\ &\text{-}\omega\mathbf{v}_1=\left(\mathbf{k}\times\mathbf{B_1}\right)\times\frac{\mathbf{B_o}}{\rho_{mo}\mu_o}-\mathbf{k}\cdot\frac{p_1}{\rho_{mo}} \\ &\text{-}\omega\mathbf{B_1}=\mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{B_o}\right) \\ &p_1\text{-}c_s^2\rho_{m1}=0 \end{aligned}\right\}$$

MHD Waves (III)

We calculate the dispersion relation:

$$\begin{cases} -\omega\rho_{m1} + \rho_{mo}\mathbf{k}\cdot\mathbf{v}_1 = 0 \\ -\omega\mathbf{v}_1 = \left(\mathbf{k}\times\mathbf{B_1}\right)\times\dfrac{\mathbf{B_o}}{\rho_{mo}\mu_o} - \mathbf{k}\cdot\dfrac{p_1}{\rho_{mo}} \\ -\omega\mathbf{B_1} = \mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{B_o}\right) \\ p_1 - c_s^2\rho_{m1} = 0 \Leftrightarrow p_1 = c_s^2\rho_{m1} \end{cases} \Rightarrow \begin{cases} \dfrac{\rho_{m1}}{\rho_{mo}} = \dfrac{\mathbf{k}\cdot\mathbf{v}_1}{\omega} \\ -\omega\mathbf{v}_1 = \left(\mathbf{k}\times\mathbf{B_1}\right)\times\dfrac{\mathbf{B_o}}{\rho_{mo}\mu_o} - \mathbf{k}\cdot c_s^2\dfrac{\rho_{m1}}{\rho_{mo}} \\ -\omega\mathbf{B_1} = \mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{B_o}\right) \end{cases} \Rightarrow$$

$$\begin{cases} -\omega\mathbf{v}_1 = \left(\mathbf{k}\times\mathbf{B_1}\right)\times\dfrac{\mathbf{B_o}}{\rho_{mo}\mu_o} - \mathbf{k}\cdot c_s^2\dfrac{\mathbf{k}\cdot\mathbf{v}_1}{\omega} \\ \mathbf{B_1} = -\dfrac{\mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{B_o}\right)}{\omega}, \text{we set } \mathbf{V_A} = \dfrac{\mathbf{B_o}}{\sqrt{\rho_{mo}\mu_o}} \end{cases} \Rightarrow \begin{cases} \omega^2\mathbf{v}_1 = \left(\mathbf{k}\times\dfrac{\mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{B_o}\right)}{\sqrt{\rho_{mo}\mu_o}}\right)\times\dfrac{\mathbf{B_o}}{\sqrt{\rho_{mo}\mu_o}} + c_s^2\mathbf{k}\cdot\left(\mathbf{k}\cdot\mathbf{v}_1\right) \\ \omega^2\mathbf{v}_1 = \left(\mathbf{k}\times\mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{V_A}\right)\right)\times\mathbf{V_A} + c_s^2\mathbf{k}\cdot\left(\mathbf{k}\cdot\mathbf{v}_1\right) \end{cases}$$

MHD Waves (IV)

Dispersion relation of MHD waves: The special case of **k** parallel to $\mathbf{B_o}$ (both along z).

$$\left.\begin{aligned}&\omega^2\mathbf{v}_1=\left(\mathbf{k}\times\mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{V_A}\right)\right)\times\mathbf{V_A}+c_s^2\mathbf{k}\cdot\left(\mathbf{k}\cdot\mathbf{v}_1\right)\\&\text{we have: }\mathbf{k}=k\hat{\mathbf{z}},\ \mathbf{V_A}=V_A\hat{\mathbf{z}}\end{aligned}\right\}\Rightarrow$$

$$\omega^2\mathbf{v}_1-k^2c_s^2v_{1z}\hat{\mathbf{z}}+\left(kV_Av_{1z}\hat{\mathbf{z}}-\mathbf{v}_1kV_A\right)kV_A=0\Rightarrow$$

$$\Rightarrow\begin{Bmatrix}\mathbf{v}_{\mathbf{1}\perp}=v_{1x}\mathbf{x}+v_{1y}\mathbf{y}\\\left(\omega^2-k^2c_s^2\right)v_{1z}=0\\\left(\omega^2-k^2V_A^2\right)\mathbf{v}_{\mathbf{1}\perp}=0\end{Bmatrix}$$

The dispersion relation indicates that there are two kinds of waves propagating along the magnetic field in MHD: A longitudinal wave from the 1st dispersion relation which describes ordinary sound waves and a transverse wave from the 2nd dispersion relation which describes the shear Alfvèn wave.

MHD Waves (IV)

Dispersion relation of MHD waves: The special case of **k** perpendicular to $\mathbf{B_o}$

$$\left.\begin{aligned}&\omega^2\mathbf{v}_1=\left(\mathbf{k}\times\mathbf{k}\times\left(\mathbf{v}_1\times\mathbf{V_A}\right)\right)\times\mathbf{V_A}+c_s^2\mathbf{k}\cdot\left(\mathbf{k}\cdot\mathbf{v}_1\right)\\&\text{for: }\mathbf{k}\perp\mathbf{B_o}\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow\left[\omega^2-\left(c_s^2+V_A^2\right)k^2\right]\mathbf{v}_1=0$$

The dispersion relation describes Magnetosonic Waves.