Introduction to plasma

Part 03: Magneto-hydrodynamics

Magnetohydrodynamics: plasma as a conducting fluid

Historically Magnetohydrodynamics, abbreviated **MHD**, preceded the development of modern plasma physics. The original intent of MHD was to treat a plasma as a **conducting fluid**. The governing equations (**Single Fluid Equations**) were adapted from fluid mechanics with appropriate modifications to account for electrical forces.

This was accomplished by using a linear Ohm's law, such as is often used to describe conducting media. Since, to a first approximation, plasmas are electrically neutral, the net charge density was assumed to be identically zero. Also, since fluid motions tend to be slow compared to the characteristic time scales of a plasma, the **displacement current was assumed to be small compared to the conduction current**. These assumptions, together with an appropriate equation of state, were sufficient to obtain a closed system of equations

MHD: Single Fluid Equations

The total mass density of plasma $\boldsymbol{\rho_m}$, the flow velocity of plasma $\mathbf{v}$, the electric charge density $\boldsymbol{\rho_q}$ and the current density $\mathbf{j}$ are defined as follows:

$$n_e \approx n_i : \text{Quasi Neutrality}$$

$$\rho_m = n_e m_e + n_i m_i \approx (\text{for p-e}^- \text{ plasma})\ n_o m_i$$

$$\rho_q = -n_e e + Z n_i e = (\text{for p-e}^- \text{ plasma})\ e\left(n_i - n_e\right)$$

$$\mathbf{v} = \frac{n_e m_e \mathbf{v_e} + n_i m_i \mathbf{v_i}}{n_e m_e + n_i m_i} = \frac{n_e m_e v_e + n_i m_i \mathbf{v_i}}{\rho_m}\ (\text{Center of Mass Velocity})$$

$$\mathbf{j} = -n_e e \mathbf{v_e} + Z n_i e \mathbf{v_i} = (\text{for p-e}^- \text{ plasma})\ e\left(\mathbf{v_i} n_i - \mathbf{v_e} n_e\right) \approx e n_o\left(\mathbf{v_i} - \mathbf{v_e}\right)$$

MHD: Single Fluid Equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m v) = 0\text{: Mass Conservation}$$

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot (\rho_q v) = 0\text{: Charge Conservation}$$

Mass and Charge Conservation: The mass conservation equation, above, is identical to the well-known mass continuity equation of fluid dynamics; the fluid velocity is the mass weighted average of the flow velocities of the individual species (electron and ions). The same argument holds for the charge conservation equation taking into account the charge (instead of mass) flow speed.

MHD: Single Fluid Equations

$$\rho_m \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \rho_q \mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p \text{ : Momentum}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma} \text{ : Ohm's Law}$$

Momentum Equation: The left side of the equation represents the rate of change of the total momentum density. The right side is the total force applied per unit volume: The first term is the **force** of the **electric field**; this term is generally dropped because it is much smaller than the **j**x**B** term. The latter represents the **Magnetic Force**. The last term is the **pressure gradient**.

Generalized Ohm's law:. It describes the electrical properties of the conducting fluid (σ is the plasma conductivity).

MHD: Single Fluid Equations

$$p\rho_m^{-\gamma} = \text{const} \Leftrightarrow \frac{d}{dt}\left(\frac{p}{\rho_m^{\gamma}}\right) = 0$$

The equation of state: Since the moment equations do not define a closed system of equations, we must choose an equation of state in order to close the system of equations. The equation of state specifies the plasma pressure as a function of the temperature and density. The equation of state is commonly assumed to be a power law (see above). The exponent **γ** is called the polytrope index. By choosing various values for **γ** , a variety of situations can be represented. For example: For isothermal processes **γ =1** we have while for adiabatic **γ =(n+2)/n** which is **5/3** for monoatomic gas.

MHD: All Single Fluid Equations

$$\frac{\partial \rho_m}{\partial t} + \nabla\left(\rho_m \mathbf{v}\right) = 0\text{: Mass Conservation}$$

$$\frac{\partial \rho_q}{\partial t} + \nabla\left(\rho_q \mathbf{v}\right) = 0\text{: Charge Conservation}$$

$$\rho_m\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \rho_q\mathbf{E} + \mathbf{j}\times\mathbf{B} - \nabla p \text{ :Momentum}$$

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = \frac{\mathbf{j}}{\sigma} \text{ : Ohm's Law}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ : Faraday}$$

$$\nabla\times\mathbf{B} = \mu_o \mathbf{j} \text{ : Ampere}$$

$$p\rho_m^{-\gamma} = \text{const} \Leftrightarrow \frac{d}{dt}\left(\frac{p}{\rho_m^{\gamma}}\right) = 0 \text{ : State}$$

We have 15 equation in the 15 unknowns, **p**, $\boldsymbol{\rho_m}$, $\boldsymbol{\rho_q}$, **v**, **j**, **E** and **B**.

MHD: Magnetic Pressure

When there is a gradient in the plasma pressure, then under static equilibrium conditions there will be a gradient in the magnetic pressure. From the momentum transport equation (steady state, all time derivatives=0, and having dropped the small E.ρ_q term) and the Amber Law we have, eliminating **j**:

$$\left.\begin{array}{l}\text{Momentum: } 0=\mathbf{j}\times\mathbf{B}-\nabla p \\ \text{Ampere: } \nabla\times\mathbf{B}=\mu_o\mathbf{j}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\mathbf{j}\times\mathbf{B}=\nabla p \\ \dfrac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\mu_o}=\nabla p\end{array}\right\} \Rightarrow \mathbf{j}\times\mathbf{B}=-\nabla\frac{\mathbf{B}^2}{\mu_o}+\frac{(\mathbf{B}\cdot\nabla)}{\mu_o}\mathbf{B}=\nabla p \Rightarrow$$

$$\text{Magnetic and Thermal Pressures: } \nabla\left(p+\frac{\mathbf{B}^2}{\mu_o}\right)=\frac{(\mathbf{B}\cdot\nabla)}{\mu_o}\mathbf{B}\text{: Magnetic Tension}$$

$$\text{and } \nabla\left(p+\frac{\mathbf{B}^2}{\mu_o}\right)=0\text{: Pressure Balance when the field lines are straight and parallel.}$$

MHD: Derivation of the Induction equation

$$\left.\begin{aligned}&\text{Ohm's Law: } \mathbf{E}+\mathbf{V}\times\mathbf{B}=\frac{\mathbf{j}}{\sigma}\\ &\text{Faraday: } \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\\ &\text{Ampere : } \nabla\times\mathbf{B}=\mu_o\mathbf{j}\end{aligned}\right\}\Leftrightarrow\left\{\begin{aligned}&\mathbf{E}=\frac{\mathbf{j}}{\sigma}-\mathbf{V}\times\mathbf{B}\\ &\mathbf{E}=\frac{\nabla\times\mathbf{B}}{\sigma\mu_o}-\mathbf{V}\times\mathbf{B}\\ &\frac{\partial\mathbf{B}}{\partial t}=\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)-\nabla\times\left(\frac{\nabla\times\mathbf{B}}{\sigma\mu_o}\right)\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow\begin{cases}\dfrac{\partial\mathbf{B}}{\partial t}=\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)+\dfrac{1}{\sigma\mu_o}\nabla^2\mathbf{B}\\ \nabla\times\left(\nabla\times\mathbf{B}\right)=\nabla\left(\nabla\cdot\mathbf{B}\right)-\nabla^2\mathbf{B}\\ \text{Where } \dfrac{1}{\sigma\mu_o} \text{ is the magnetic diffusivity}\end{cases}$$

MHD: More on the Induction equation

$$\text{Induction Equation: } \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{V} \times \mathbf{B} \right) + \frac{1}{\sigma \mu_o} \nabla^2 \mathbf{B} \Rightarrow$$

$$\left\{ \begin{array}{l} \text{diffusivity: } \eta = \dfrac{1}{\sigma \mu_o} \\ \dfrac{1}{\sigma \mu_o} \nabla^2 \mathbf{B} \sim \dfrac{|\mathbf{B}|}{\sigma \mu_o L^2} \\ \nabla \times \left(\mathbf{V} \times \mathbf{B} \right) \sim \dfrac{|\mathbf{V}| \cdot |\mathbf{B}|}{L} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \text{Magnetic Reynolds Number} \\ R_m = L|\mathbf{V}|\sigma \mu_o = \dfrac{L|\mathbf{V}|}{\eta} \end{array} \right.$$

If the fluid moves with a typical speed **V** and a typical length scale **L** then, from order of magnitude calculations, we obtain the magnetic Reynolds number $\mathbf{R_m}$. The magnetic Reynolds number provides a criterion which indicates if the first term (convective) or the second (diffusive) on the right side of the equation dominates.

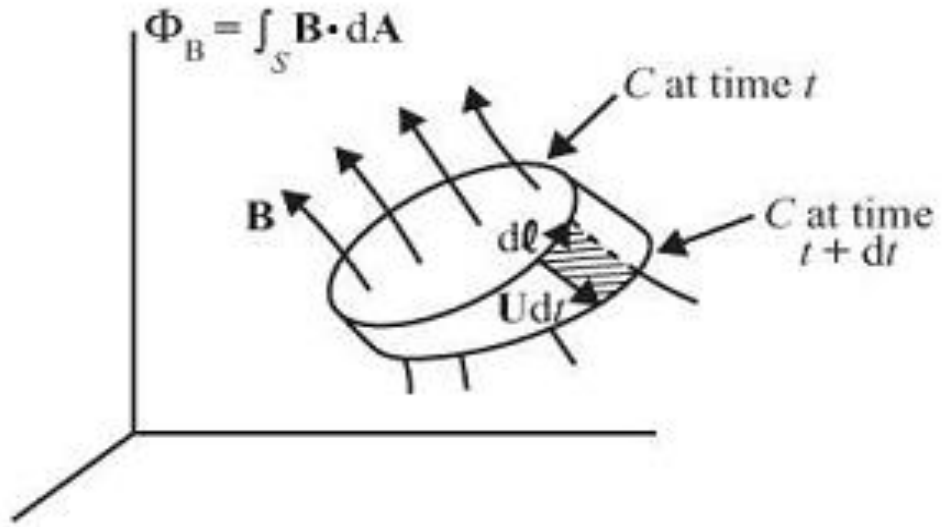
MHD: More on the Induction equation: Perfectly conducting limit

When the magnetic Reynolds number $\mathbf{R_m>>1}$ (η->0) the <u>diffusive</u> term in the induction equation vanishes. The induction equation for an ideal conductive fluid is shown below. Based on it we will demonstrate the **Alfvén's theorem**, which states *"that in a fluid with <u>infinite electric conductivity</u>, the <u>magnetic field is frozen into the fluid</u> and has to move along with it"*, Suppose that curve **C** moves with the fluid, with each point on the curve moving to a new point **Vdt** after a time **dt**. The rate of change of the magnetic flux through curve C is given by the sum of two terms The first gives the rate of change of the flux due to the explicit time dependence of B, and the second the rate of change of the flux due to the motion of the curve C. :

$$\left.\begin{array}{l}\text{Induction Equation } \mathrm{R_L} \gg 1\text{: } \dfrac{\partial \mathbf{B}}{\partial \mathrm{t}} = \nabla \times (\mathbf{V} \times \mathbf{B}) \\ \text{Change of } \Phi_{\mathrm{B}} \text{ over time: } \dfrac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{dt}} = \displaystyle\int_{S} \dfrac{\mathrm{d}\mathbf{B}}{\mathrm{dt}} \mathrm{d}\mathbf{S} + \oint_{C} \mathbf{B} \cdot \mathbf{V} \times \mathrm{d}\mathbf{l}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l}\dfrac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{dt}} = \displaystyle\int_{S} \nabla \times (\mathbf{V} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S} + \oint_{C} \mathbf{B} \cdot \mathbf{V} \times \mathrm{d}\mathbf{l} \\ \text{using Stokes' theorem}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{dt}} = \oint_{C} \mathbf{V} \cdot \mathbf{B} \times \mathrm{d}\mathbf{l} + \oint_{C} \mathbf{B} \cdot \mathbf{V} \times \mathrm{d}\mathbf{l} = 0$$



MHD: More on the Induction equation: Diffusive limit

When the magnetic Reynolds number $\mathbf{R_m \ll 1}$ ($\sigma \to 0$) , the diffusive term overcomes the convective term. In an electrically resistive fluid with large values of **η**, the magnetic field is diffused away very fast, and the Alfvén's Theorem cannot be applied.The magnetic energy is dissipated to heat and other types of energy. The induction equation becomes:

$$\text{Induction Equation } R_m \ll 1: \ \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\sigma \mu_o} \nabla^2 \mathbf{B} \Rightarrow$$

$$\text{To Order of Magnitude: } \Rightarrow \begin{Bmatrix} \dfrac{\partial \mathbf{B}}{\partial t} \approx \dfrac{\mathbf{B}}{\tau_d} \\ \dfrac{1}{\sigma \mu_o} \nabla^2 \mathbf{B} \approx \dfrac{1}{\sigma \mu_o} \dfrac{\mathbf{B}}{L^2} \end{Bmatrix} \Rightarrow \ \tau_d = \sigma \mu_o L^2 = \frac{L^2}{\eta}$$

The dissipation time scale $\boldsymbol{\tau_d}$ is the time scale for the dissipation of magnetic energy over a length scale **L.**